EXPONENTIAL CONVERGENCE OF THE TRACKING ERROR IN ADAPTIVE SYSTEMS WITHOUT PERSISTENT EXCITATION

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Abstract

Conditions are investigated for exponential convergence of the tracking error in feedforward adaptive systems without persistent excitation. Particular attention is paid to the continuous-time LMS algorithm in the overparametrized case. A main result is that for a bounded periodic regressor, the tracking error converges exponentially without regard to parameter convergence or to the degree of overparametrization. These results remove the persistent excitation (PE) conditions and parameter convergence conditions previously thought necessary to ensure exponential tracking error convergence in this class of systems,

1 Introduction

Persistent excitation (PE) conditions which ensure parameter convergence in adaptive algorithms have been studied by many researchers. Early results can be found in Astrom and Bohlin [I] where the PE condition is expressed in terms of positive definiteness of the autocorrelation function formed from the regressor, Subsequently, Bitmead and Anderson [4] proved that parameter convergence is *exponential* when PE conditions are satisfied in the least-mean square (LMS) algorithm and the normalized LMS algorithms. Explicit upper and lower bounds on the exponential response can be found in [1 1]. A general discussion of the PE condition is given in [3] and an effort to unify many definitions can be found in [13].

One important consequence of exponential parameter convergence, is that the tracking error (which is linear in the parameter error) also converges exponentially, This relationship gives the (false) impression that exponential tracking error convergence requires the same stringent PE conditions as parameter convergence. Interestingly, there are several

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indications to the contrary. For example, using an approximate linear analysis, Glover [6] indicated as early as 1977 that exponential convergence of the tracking error is possible in the feedforward adaptive LMS algorithm with tap delay line basis functions, and sinusoidal excitation, without any conditions on parameter convergence,

Motivated by Glover, this paper investigates exponential convergence of the tracking error, without regard to parameter convergence. A main result of the paper is that for a bounded periodic regressor, the tracking error converges exponentially without any PE condition. This result is important, because in many applications good tracking performance is required while it is not desirable or even possible to satisfy PE conditions [10].

In Section 2 the LMS adaptive algorithm is reviewed in a continuous-time setting, Convergence properties are reviewed in Section 3 and new results pertaining to exponential tracking error convergence are presented. In Section 4, exponential convergence results are specialized to the important case of a tap-delay line basis. The analytic convergence bounds are validated by a simulation example, and arc shown to be consistent with earlier exponential convergence bounds of Glover [6].

2 LMS Algorithm

In this section, the Least Mean Square (LMS) algorithm [12] is reviewed. The algorithm is presented in a continuous-time framework, consistent with the treatment found in [9].

Let the $y(t) \in R^1$ and $x(t) \in R^N$, be known signals and assume there exists a constant parameter vector $w^o \in R^N$ such that,

$$y(t) = w^{oT} x(t) \tag{1}$$

for all $t > 0$. Uniqueness of w^o is not required (i.e., the system can be overparametrized). An estimate $\hat{y}$ of y is constructed as,

$$\hat{y} = w(t)^T x(t) \tag{2}$$

where $w(t)$ is an estimate of w^o, tuned in real-time using the continuous-time LMS algorithm,

$$\dot{w} = \mu x(t) e(t) \tag{3}$$

with adaptation gain $\mu > 0$. The tracking error is defined as,

$$e(t) = y(t) - \hat{y}(t) \tag{4}$$

and the parameter error is defined as,

$$\phi(t) = w^o - w(t) \tag{5}$$

Using (l)(2)(4)(5), the tracking and parameter errors can be related as follows,

$$e = \phi^T x(t) \tag{6}$$

Assuming that the true parameter w^o does not vary with time, (i. e., $\dot{w}^o = 0$), it follows from (3)(5) that,

$$\dot{\phi} = \dot{w}^o - \dot{W} = -\mu x e \tag{7}$$

This equation characterizes the propagation of the parameter error.

3 Exponential Convergence Properties

It is convenient at this point to review a well-known stability argument. Define the Lyapunov function candidate,

$$V = \frac{1}{2}\phi^T\phi \tag{8}$$

Taking the derivative of (8) and using (l)-(7) yields,

$$\dot{V} = -\mu e \phi^T x = -\mu e^2 \leq 0 \tag{9}$$

This proves that ϕ remains bounded. If x is bounded, then from (6) the error e remains bounded. Furthermore, if $\dot{x}$ is bounded, then $\ddot{V}$ is bounded, $\dot{V}$ is uniformly continuous, and Barbalat's lemma ([9], pg. 85, and 276), can be applied to ensure that $\lim_{t\to\infty} e = 0$. This well known argument ensures that the error converges to zero as desired.

While the above argument ensures that e converges to zero, it does not indicate *how fast* it converges. Additional conditions can be imposed which ensure exponential convergence of e to zero, For example, if $x(t)$ is a periodic function with period T_o, then it is well known (cf., [9]), that the existence of some $\beta > 0$ such that,

$$\beta \cdot I \leq M \tag{10}$$

$$M = \frac{1}{T_o}\int_0^{T_o} x(t)x(t)^T\, dt \tag{11}$$

ensures that the error converges exponentially with rate β, i.e., there exists a $c_0 \geq 0$ such that,

$$|e| \leq c_0 e^{-\mu\beta t} \tag{12}$$

It is shown in this paper that the PE condition (10) is overly restrictive, Specifically, the main result proves that for x bounded and periodic, the convergence of e to zero is generically exponential *without any* condition on *M*, This result is most useful in applications where performance is measured based on the convergence properties of e rather than on ϕ. It will be seen that in the overparametrized case where *M* is singular, the tracking error still converges exponentially even though the PE condition (10) is not satisfied.

THEOREM 1 Assume there exists a w^o such that (1) holds for all $t \geq 0$, and that the LMS algorithm (2)-(7) is used to tune w. Let $x(t) \in R^N$ be a bounded periodic function of $t \geq 0$, with period T_o, i.e.,

$$||x(t)|| \leq \eta < \infty; \text{ for all } t \geq 0 \tag{13}$$

$$x(t + T_o) = x(t); \text{ for all } t \geq 0 \tag{14}$$

Then the error e in (6) converges to zero exponentially as,

$$|e(t)| \leq \eta e^{\alpha T_o} ||\phi(0)|| e^{-\alpha t} \tag{15}$$

$$\alpha \triangleq \mu \lambda_n(M) \tag{16}$$

$$M \triangleq \frac{1}{T_o} \int_0^{T_o} x(t) x^T(t) dt \tag{17}$$

where λ_n denotes the smallest *positive eigenvalue* of the symmetric non-negative definite matrix $M = M^T \geq 0$. Such a smallest positive eigenvalue always exists unless $x \equiv 0$, $e \equiv 0$, i.e., the regressor x and error e vanish identically.

PROOF: First consider the trivial case where $M = 0$. Then from (17), $x = 0$, and from (6) the error e vanishes identically.

Now consider the nontrivial case where $M \neq 0$. From (6)(7), one has,

$$\dot{\phi} = -\mu x e = -\mu x x^T \phi = A(t)\phi \tag{18}$$

where $A(t) = -\mu x x^T$. Equation (18) is a time-varying linear system, with the matrix-exponential solution [5],

$$\phi(t) = e^{\int_0^t A(\tau) d\tau} \phi(0) \tag{19}$$

Since x is periodic with period T_o it follows that,

$$M = \frac{1}{T_o} \int_0^{T_o} x x^T d\tau = P \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & 0 \end{bmatrix} P^T \tag{20}$$

$$\Lambda_{11} = diag\{\lambda_1, ..., \lambda_n\} > 0 \tag{21}$$

where $P^T = P^{-1}$, and, $\lambda_1 \geq \ldots \geq \lambda_n > 0$. Note. that λ_n is the smallest "nonzero" eigenvalue of M, and that it will always exist since $M \neq 0$. Define the change of variable,

$$z = P^T x \tag{22}$$

Then,

$$\frac{1}{T_o} \int_0^{T_o} z z^T d\tau = P^T \left(\frac{1}{T_o} \int_0^{T_o} x x^T d\tau\right) P = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & 0 \end{bmatrix} \tag{23}$$

Let z be partitioned as,

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \tag{24}$$

where $z_1 \in R^n$ and $z_2 \in R^{N-n}$, Then it follows from (23) that,

$$\frac{1}{T_o}\int_0^{T_o} z_1 z_1^T d\tau = \Lambda_{11} > 0 \tag{25}$$

and,

$$z_2(t) \equiv 0 \tag{26}$$

Hence, z(t) is of the form,

$$z(t) = \begin{bmatrix} z_1(t) \\ 0 \end{bmatrix} \tag{27}$$

Define,

$$r(t) \triangleq S^T P^T \phi(t) \tag{28}$$

$$S = \begin{bmatrix} I_{nn} \\ \mathcal{O} \end{bmatrix} \in R^{N\times n} \tag{29}$$

where I_{nn} is the $n \times n$ identity matrix and $\mathcal{O} \in R^{(N-n)\times(N-n)}$ is a matrix containing all zeros, It is seen that the matrix S simply "sifts out" the first n elements of $P^T\phi$ for inclusion into r.

Writing the error equation (6) in terms of the reduced vectors z_1 in (27) and r in (28) yields,

$$e = \phi^T x = \phi^T P P^T x = \phi^T P z = \phi^T P S z_1 = r^T z_1 \tag{30}$$

Similarly, the adaptation law (7) can be written in terms of the reduced vectors as,

$$\dot{r} = S^T P^T \dot{\phi} = -\mu S^T P^T x e = -\mu S^T z e = -\mu z_1 e \tag{31}$$

Expanding the adaptation law (31), using (30) gives,

$$\dot{r} = -\mu z_1 e = -\mu z_1 z_1^T r \tag{32}$$

Using (25), equation (32) is seen as a time-varying linear system with matrix exponential solution [5],

$$r(kT_o) = e^{-\mu\int_0^{kT_o} z_1 z_1^T d\tau} r(0) \tag{33}$$

$$= e^{-\mu k T_o \Lambda_{11}} r(0) = diag\{e^{-\mu\lambda_1 kT_o}, \dots, e^{-\mu\lambda_n kT_o}\} r(0) \tag{34}$$

Using (13)(34) in (30) yields,

$$|e(kT_o)| = |r(kT_o)^T z_1(kT_o)| = |r(0)^T e^{-\mu k T_o \Lambda_{11}} z_1(kT_o)| \leq \eta ||r(0)|| e^{-\mu\lambda_n kT_o} \tag{35}$$

This implies that the error dies exponentially on the discrete-time grid kT_o, $k = 0,.,,, \infty$. It only remains to show that the convergence is exponential on the continuous-time interval $t \in$ [0,00).

It follows from (32) that $||r(t)||$ is nonincreasing. Hence, using (34) one has,

$$||r(t)|| \leq ||r(kT_o)|| \leq e^{-\alpha k T_o} ||r(0)|| \quad \text{for } t \in [kT_o, (k+1)T_o] \tag{36}$$

where $\alpha \triangleq \mu\lambda_n$. The inequality $t \leq (k+1)T_o$ also implies that $t - T_o \leq kT_o$, which leads to the relation,

$$e^{-\alpha k T_o} \leq e^{-\alpha(t-T_o)} = e^{\alpha T_o} e^{-\alpha t} \quad \text{for } t \in [kT_o, (k \text{ -I- } 1)T_o] \tag{37}$$

Combining (36) and (37) gives,

$$||r(t)|| \leq e^{\alpha T_o} ||r(0)|| e^{-\alpha t} \tag{38}$$

The following bounds will also be useful,

$$||r(0)|| = ||S^T P^T \phi(0)|| \leq ||\phi(0)|| \tag{39}$$

$$||z_1|| = ||z|| = ||x|| \leq \eta \tag{40}$$

Substituting (38)(39)(40) into (30) yields upon rearranging,

$$|e(t)| = |r^T z_1| \leq ||r|| \cdot ||z_1|| \leq \eta e^{\alpha T_o} ||\phi(0)|| e^{-\alpha t} \tag{41}$$

■

REMARK 1 Theorem 1 says that for bounded periodic x, the error e is either identically zero, or it converges to zero exponentially fast. There are no other possibilities. Interestingly, the result is independent of the number of parameters IV, as long as a solution w^o exists in (1). Note that the usual PE requirement for at least N/2 sinusoids in the regressor is avoided.

REMARK 2 Persistent excitation conditions are e] iminated in Theorem 1 by avoiding the need for convergence of the full parameter vector w in the proof, Rather, the '(degree" to which the given regressor x is persistently exciting is indicated by the number n of nonzero eigenvalues of M. This is used to define an appropriately reduced parameter error vector r $\in R^n$. It can be seen from (38) that the reduced vector r converges exponentially, which from (41) ensures exponential convergence of e.

4 Tap-Delay Line Basis

In this section, the results of Section 3 are specialized to the case of a tap delay line basis.

Let the components of the regressor $x = [x_1, ..., x_N]^T \in RN$ be defined by filtering a signal $\xi(t) \in R^1$ through a tap delay line with N taps and tap delay T, i.e.,

$$x_\ell = e^{-(\ell-1)sT}\xi, \quad \ell = 1, ..., N \tag{42}$$

Let the measured signal ξ be given by the following sum of m sinusoids,

$$\xi(t) = \sum_{i=1}^{m} A_i sin(\omega_i t + \phi_i) \tag{43}$$

Note that (43) represents a slight generalization of the. earlier formulation to include aperiodic signals, As such, the following general definition for the covariance will be used,

$$\overline{M} = \lim_{T_o \to \infty} \frac{1}{T_o} \int_0^{T_o} x(t)x(t)^T dt \tag{44}$$

For the special case of periodic signals, $\overline{M}$ in (44) and M in (17) are identical.

By direct integration, the correlation matrix $\overline{M}$ (44) can be calculated in closed form as,

$$\overline{M} = \sum_{i=1}^{m} \frac{A_i^2}{2} \begin{bmatrix} 1 & cos\omega_i T & cos\ 2\omega_i T & \ldots & cos(N-1)\omega_i T \\ cos\omega_i T & 1 & cos\omega_i T & \ldots & cos(N-2)\omega_i T \\ \vdots & \ddots & \ddots & \ddots & \\ cos(N-1)\omega_i T & cos(N-2)\omega_i T & \ldots & cos\omega_i T & 1 \end{bmatrix} \tag{45}$$

The following factorization of $\overline{M}$ will be useful.

LEMMA 1 *(Outer-Product Factorization)* Let the regressor $x = [x_1, \ldots, x_N]^T \in RN$ in the LMS algorithm (l)-(7) be obtained by filtering $\xi \in R^1$ through the tap delay line (42), with N taps and tap delay T. Let ξ be given as the sum of sinusoids,

$$\xi(t) = \sum_{i=1}^{m} A_i sin(\omega_i t + \phi_i) \tag{46}$$

where,

$$\begin{aligned} m &\leq N/2 & & \\ 0 &< A_i < \infty & &\text{for } i = 1, ..., m \\ 0 &< \omega_i T < \pi, & &\text{for } i = 1, ..., m \\ \omega_i &\neq \omega_j, & &\text{for } i \neq j \end{aligned} \tag{47}$$

Then,

(i) The covariance $\overline{M}$ defined in (44) can be written in terms of the outer-product factorization,

$$\overline{M} = FF^T \in R^{N \times N} \tag{48}$$

where the square-root factor F is given by,

$$F = CA \in R^{N \times 2m} \tag{49}$$

$$C \triangleq [c_1, s_1, \ldots, c_m, s_m] \in R^{N \times 2m} \tag{50}$$

$$c_i \triangleq \begin{bmatrix} 1 \\ cos\omega_i T \\ \vdots \\ cos(N-1)\omega_i T \end{bmatrix} \in R^N; \qquad s_i \triangleq \begin{bmatrix} 1 \\ sin\omega_i T \\ \vdots \\ sin(N-1)\omega_i T \end{bmatrix} \in RN \tag{51}$$

$$A \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} A_1 \cdot I_{22} & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & A_m \cdot 1_{22} \end{bmatrix} \in R^{2m \times 2m} \tag{52}$$

where 1_{22} denotes the 2 x 2 identity matrix,

(ii) The rank of $\overline{M}, C$ and F are the same and equal to $2m$.

PROOF: Because of the Toeplitz structure of $\cancel{M}$, the $\{j, k\}$th entry of the ith term in the sum (45), denoted as $\overline{M}(i)_{jk}$, can be written as

$$\begin{aligned} \overline{M}(i)_{jk} &= cos(|j - k|\omega_i T) = cos((j-1)\omega_i - (k-1)\omega_i) \\ &= cos((j-1)\omega_i)cos((k-1)\omega_i) + sin((j-1)\omega_i)sin((k-1)\omega_i) \end{aligned} \tag{53}$$

Substituting (53) into (45) gives upon rearranging,

$$\overline{M} = \sum_{i=1}^{m} \frac{A_i^2}{2}(c_i c_i^T + s_i s_i^T) \tag{54}$$

The vector outer-product sum (54) can be put into matrix form ass,

$$\overline{M} = CAAC^T = FF^T \tag{55}$$

which proves (i).

Let C_s be the matrix constructed from the top $2m$ rows of C. Then,

$$C_s C_s^T = \begin{bmatrix} 1 & cos\omega_i T & \cdots & cos(2m-1)\omega_i T \\ cos\omega_i T & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & cos\omega_i T \\ cos(2m-1)\omega_i T & \cdots & cos\omega_i T & 1 \end{bmatrix} \in R^{2m\times 2m} \tag{56}$$

The matrix (56) is found to arise in optimal experiment design, and has the known determinant (cf., [8], page 140),

$$det\{C_s C_s^T\} = 4^{m(m-1)} \prod_{k=1}^{m} sin^2(\omega_k T) \prod_{i=1}^{m-1} \prod_{j=i+1}^{m} (cos(\omega_i T) - cos(\omega_j T))^4 \tag{57}$$

For distinct frequencies $\omega_i \neq \omega_j$, and $0 < \omega_i T < \pi$, monotonicity of $cos(\nu)$ over the interval $0 < \nu < \pi$ ensures that the determinant (57) does not vanish, and that C_s is nonsingular. It can be concluded that rank of C is full (i.e., equal to $2m$) since its top 2m rows forms the square nonsingular matrix C_s. Furthermore, $A_i > 0, i = 1, ..., m$ by assumption, so that $A > 0$ is nonsingular. Hence, the rank of C and $F = CA$ are identical and equal to $2m$. Finally, it can be concluded that the rank of $\bar{M}$ is $2m$ since by (55) it has F as a full rank outer-product factor. This proves (ii). ■

Lemma 1 implies that $\bar{M}$ (which is of size. N x N), will have rank $2m$. For the case where 2m = N, the covariance $\bar{M}$ is nonsingular and the usual PE conditions are satisfied to ensure exponential convergence of both ϕ and e to zero, In contrast, in the overparametrized case where 2m < N, the covariance $\bar{M}$ is singular, the PE conditions no longer hold, and ϕ no longer converges exponentially. However, one may still resort to Theorem 1 to conclude exponential convergence of e. The overparametrized case will be treated along these lines in the next result.

THEOREM 2 Let $x \in R^N, \xi \in R^1$ be as defined in Lemma 1 using the LMS algorithm (1)-(7) and the tap delay line basis (42) with N taps and tap delay T. Furthermore, let $\xi(t)$ be periodic with period T_o. Then the tracking error e converges exponentially fast,

$$|e(t)| \leq \eta e^{\alpha T_o} ||\phi(0)|| e^{-\alpha t} \tag{58}$$

with rate,

$$\alpha = \mu \lambda_{min}(F^T F) > 0 \tag{59}$$

where λ_{min} denotes the smallest eigenvalue of the "inner product" $F^T F$, and $F = CA$ *is* the square-root factor of $M = \bar{M}$ defined in Lemma 1.

PROOF: The regressor $x(t)$ is periodic since it is obtained in (42) by filtering $\xi(t)$ which is periodic, For periodic x, the covariance M in (17) is equivalent to $\bar{M}$ in (44). Hence, from

Lemma 1, result (ii), the covariance M has rank $2m$. Accordingly, the smallest nonzero eigenvalue of M is $\lambda_{2m}(M)$, where $\lambda_1 \geq \ldots \geq \lambda_{2m} > 0$ denotes the nonzero eigenvalues ordered by size.

Since the regressor $x(t)$ is bounded and periodic, it satisfies the conditions of Theorem 1. Hence, an exponential bound on the error is given as,

$$|e(t)| \leq \eta e^{\alpha T_o} ||\phi(0)|| e^{-\alpha t} \tag{60}$$

$$\alpha \triangleq \mu \lambda_{2m}(M) \tag{61}$$

It is known from Lemma 1 that M admits the outer-product factorization,

$$M = FF^T \tag{62}$$

However, it is also known (cf., [7]), that the nonzero eigenvalues of a symmetric inner product $F^T F$ and symmetric outer product FF^T *are* the same, Hence,

$$\lambda_{2m}(M) = \lambda_{2m}(FF^T) = \lambda_{2m}(F^T F) = \lambda_{min}(F^T F) > 0 \tag{63}$$

Using (63) in (60)(61) gives (58)(59) as desired, ■

From the above analysis, it is worth noting that (for the tap-delay line basis), positive definiteness of the *outer product* $M = FF^T$ *is* required for exponential convergence of both ϕ and e (i.e., the usual PE condition), while only positive definiteness of the inner *product* $F^T F$ *is* required for exponential convergence of e. The inner product condition is weaker (i.e., easier to satisfy) since $2m < N$ without PE, and F will be a "tall" matrix. Theorem 2 gives specific conditions for which the inner product condition is satisfied, and provides bounds on the exponential convergence rate.

REMARK 3 As in Theorem 1, the exponential convergence argument in Theorem 2 is independent of the number of sinusoids m in the excitation signal ξ. A colorful example will be given in Section 5 where 50 taps are used to track 2 sinusoids.

REMARK 4 In Lemma 1 and Theorem 2, the condition that the frequencies ω_i in the excitation signal ξ be less than π/T, is required to ensure that the rank of M is exactly $2m$. However, this condition can be thought of in terms of the usual Nyquist Sampling Theorem, where the tap delay T plays the role of a sampling period, and the effective sampling frequency $2\pi/T$ must exceed any signal frequency ω_i by at least a factor of 2.

5 Numerical Example

An example is given in this section demonstrating exponential tracking error convergence without persistent excitation. Specifically, N = 50 parameters (i.e., taps) will be used to track $m = 2$ sinusoids. (Note that to ensure parameter convergence, the PE condition (10) would require at least 25 sinusoids).

Let y be a sum of two tones,

$$y = sin(\omega_1 t) + sin(\omega_2 t) \tag{64}$$

where $\omega_1 = 2\pi \cdot 25$, $\omega_2 = 2\pi \cdot 50$, and let the excitation signal ξ be given as,

$$\xi = A_1 sin(\omega_1 t - \pi/4) + A_2 sin(\omega_2 t - \pi/4) \tag{65}$$

where $A_1 = A_2 = \sqrt{2}$. Let the tap delay line basis discussed in Section 4 be heavily overparametrized with N = 50 taps and tap delay $T = .004$. The LMS algorithm (1)-(7) is used to tune the error to zero, using an adaptive gain of $\mu = .1$. A simulation of the response is shown as the solid line in Figure 1.

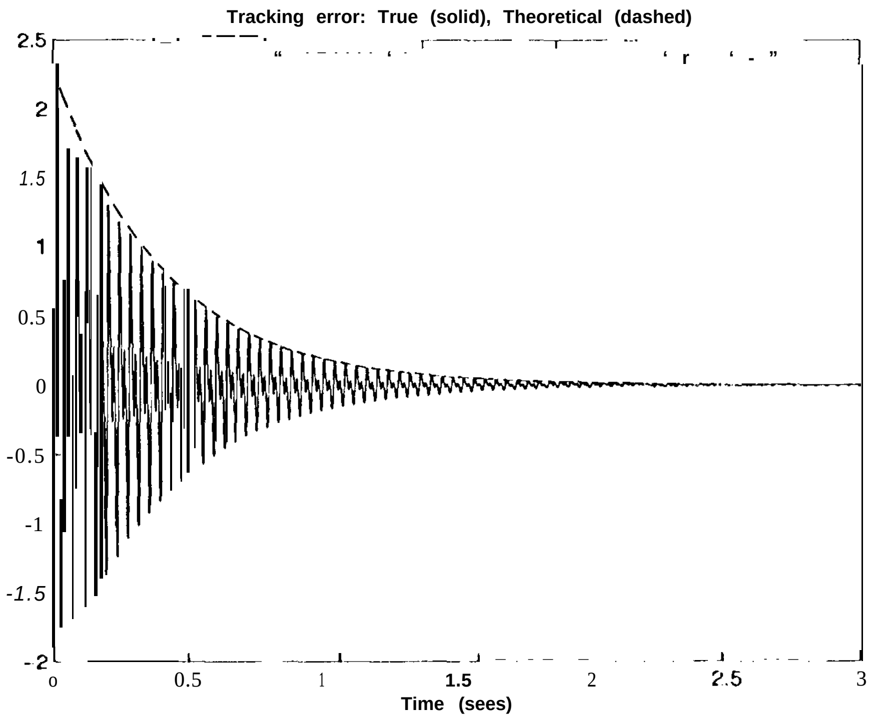

Figure 1: Exponential tracking error response from LMS algorithm with heavily overparametrized tap delay line (N = 50 taps and m = 2 sinusoids).

The exponential convergence rate from Theorem 2 is calculated as $\alpha = 2.5000$, and the theoretical bound on e is shown as the dotted line in Figure 1. It is seen that the theoretical exponential overbound on the response is quite accurate.

In a 1977 paper [6], Glover studies the mapping from y to e in the LMS algorithm with a tap delay line basis. Glover argues that for a sufficiently large number of taps, this mapping can be approximated as the linear time-invariant system, (specialized here to the case of two tones),

$$e = \frac{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}{(s^2 + \omega_1^2)(s^2 + \omega_2^2) + \frac{\mu N}{2}\left(A_1^2(s^3 + \omega_2^2 s) + A_2^2(s^3 + \omega_1^2 s)\right)} \cdot y \qquad (66)$$

Note the nice interpretation of (66) as a stable double notch filter at precisely the frequencies of the disturbance. For the present example the closed-loop poles are located at, $(-2.4992 \pm 314.10j), (-2.5008 \pm 157.09j)$. Hence, Glover's approximate analysis predicts an exponential convergence with a rate which is determined by the real part -2.4992 of the least damped pole, or equivalently $\alpha = 2.4992$. This value agrees quite well with the value $\alpha = 2.5000$ determined using the exact analysis of Theorem 2, which in turn agrees very well with the simulation results.

6 Conclusions

This paper considers exponential convergence of the tracking error in the feedforward adaptive LMS algorithm. It is shown (i.e., Theorem 1) that for a bounded periodic regressor, the tracking error converges exponentially without regard to parameter convergence or the degree of overparametrization.

Results are specialized to the tap-delay line basis (i.e., Theorem 2), giving rise to specific conditions for exponential convergence and bounds on the convergence rate. The new analytic convergence bounds are validated by a simulation example, and are shown to closely match the convergence rates predicted by an earlier approximate linear analysis due to Glover. Hence, the specialization of the new theory to the tap-delay line case acts to corroborate earlier evidence, and provide a precise theoretical basis for Glover's work.

The main result is relevent to applications where fast tracking error convergence is desired, Applications include adaptive vibration damping, feedforward neural networks, acoustic noise canceling, and adaptive feedforward control of narrowband signals.

In the present paper, the assumption that the regressor x is periodic is somewhat restrictive. Future efforts will be aimed at relaxing this condition to include almost-periodic functions and possibly more generalized signals such as the Yuan-Wonham class [13].

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